BGG resolutions, Koszulity, and stratifications: categorifying character formulae in $U_q^\iota(\mathfrak{sl}_2)$ using the nil-Brauer algebra

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The ι-quantum group

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- This has certain special bases.

Change of basis

Character formula (Brundan-Wang-Webster 2023, [BWW23a])

$$[L_n] = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(1+2\delta_{n\not\equiv t})}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4k})} [\overline{\Delta}_{n+2k}],$$

where $[L_n]$ is the dual canonical basis and $[\overline{\Delta}_n]$ is the dual PBW basis.

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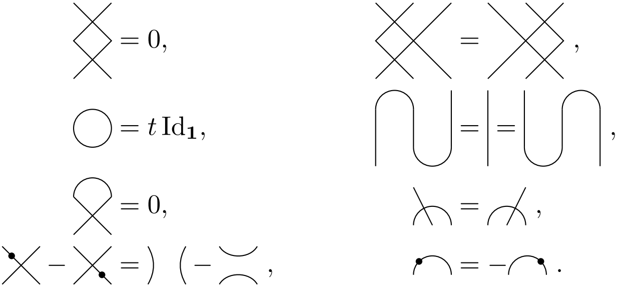
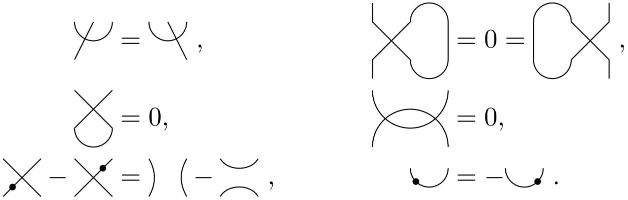
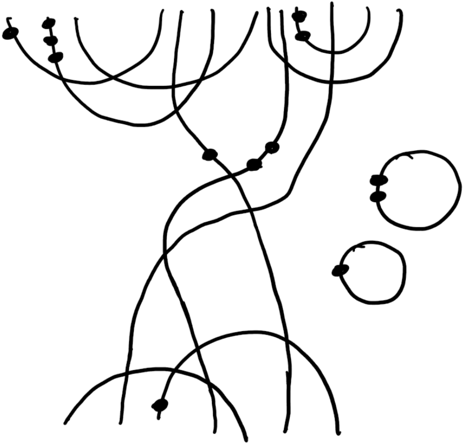


One can then show the following are also satisfied



A typical example of an element of this algebra:



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In N$\mathcal{B}$, let $\Theta = \mathbb{N}$ under $0 < 1 < \cdots$, and let e^θ be the idempotent corresponding to θ strands:

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Definition

A is "graded triangular-based" if there are (homogeneous) sets $\mathrm{X}(i,\alpha) \subseteq 1^i A 1^\alpha$, $\mathrm{H}(\alpha,\beta) \subseteq 1^\alpha A 1^\beta$, $\mathrm{Y}(\beta,j) \subseteq 1^\beta A 1^j$ such that

1. products of these elements in these sets give a basis for A, i.e.

$$\left\{ xhy : (x,h,y) \in \bigcup_{i,j,\alpha,\beta} \mathrm{X}(i,\alpha) \times \mathrm{H}(\alpha,\beta) \times \mathrm{Y}(\beta,j) \right\}$$

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3. for $\alpha \neq \beta$,

$$\begin{aligned} \mathrm{X}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) > \varpi(\beta), \\ \mathrm{H}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) = \varpi(\beta), \\ \mathrm{Y}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) < \varpi(\beta); \end{aligned}$$

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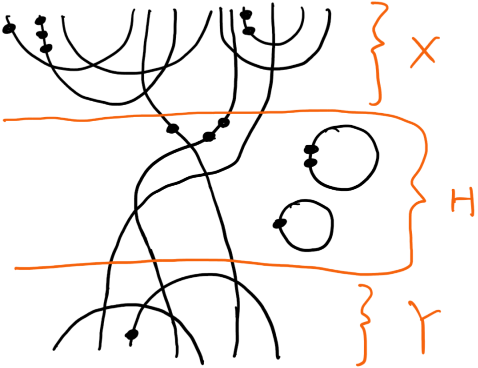
Example

$\mathrm{N}\mathcal{B}$ is graded-triangular-based by setting $\Theta = \mathbb{N}$, with e^n being the idempotent for n strands.

Same typical example from earlier:



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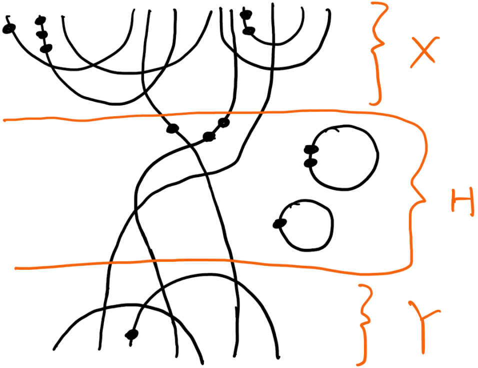

Remark: $Y(\alpha, \alpha) = \{1^\alpha\}$ means the straight lines is a Y-diagram.

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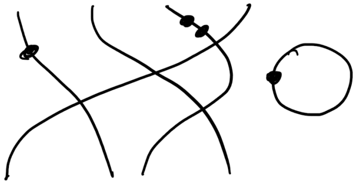

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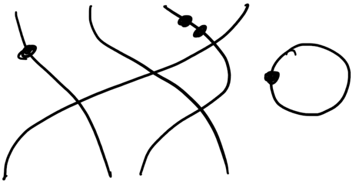

Let Λ_θ label the simples $L_\lambda(\theta)$ and projectives $P_\lambda(\theta)$ of A^θ, and let $\Lambda = \bigsqcup_\theta \Lambda_\theta$.

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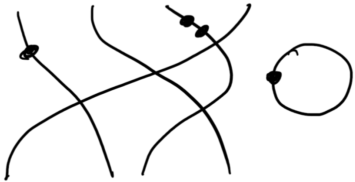

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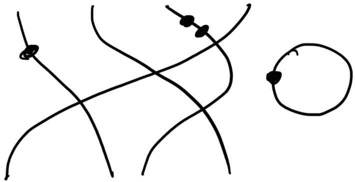

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For the nil-Brauer

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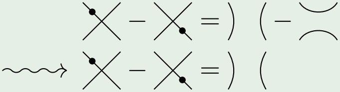
Quotienting out by $e^{\phi} : \phi < \theta$:



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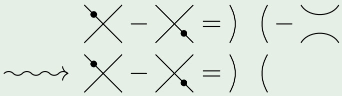

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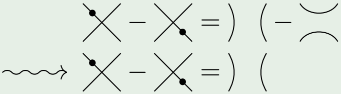

$\mathrm{N}\mathcal{B}^{\theta}$ is isomorphic to the nil-Hecke algebra on θ strands (over the ring Γ of "Schur q-functions", isomorphic to the bubbles).
This algebra has (up to grading shift) exactly one simple, so
$\Lambda = \Theta = \mathbb{N}$.

More generally

- This theory is able to handle much more, e.g. even when the idempotents corresponding to strands don't have an obvious ordering.

Standardization/costandardization

- Given a module $A^{\geq\theta} \circlearrowright M$, we can consider the functor

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Fact (Brundan 2023)

$\jmath_!^\theta$ and $\jmath_*^\theta$ are exact due to the triangular-based nature of A.

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These “(big/small) Verma modules” form an analogue of “highest-weight theory”.

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Theorem (Brundan-Wang-Webster 2023)

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Question

Can this formula be further categorified into a resolution?

The BGG resolution

Theorem (Z. 2024)

At parameter $t = 0$, the 1-dimensional simple L_0 has a BGG resolution

$$\cdots \to C^{-n}_{\mathrm{BGG}}(L_0) \longrightarrow C^{-(n-1)}_{\mathrm{BGG}}(L_0) \longrightarrow \cdots \longrightarrow C^{0}_{\mathrm{BGG}}(L_0) \longrightarrow L_0 \longrightarrow 0$$

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For other simples, we instead have a spectral sequence categorifying the character formula.

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$$\mathrm{N}\mathcal{B}^- = \bigoplus_{\psi \leq \theta} e^{\psi}\mathbb{C}\mathrm{Y}e^{\theta}.$$

The nilalgebra is Koszul

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There is a “lower-half subalgebra” $\mathrm{N}\mathcal{B}^-$ of $\mathrm{N}\mathcal{B}$ which is Koszul. This algebra is defined as

$$\mathrm{N}\mathcal{B}^- = \bigoplus_{\psi \leq \theta} e^{\psi}\mathbb{C}\mathrm{Y}e^{\theta}.$$

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This theorem is key to proving the BGG resolution.

Key ideas

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Slogan

Koszulity of half of A is intimately connected to BGG resolutions.

- Then we can use a naive resolution to compute these Ext groups.
- The spectral sequence is a resolution for modules which are Koszul over half of A.

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- We can use these tools to categorify formulae regarding Chebyshev and Hermite polynomials, using the algebras defined by Khovanov-Sazdanovic.

Algebraic recollement

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- By the $\mathsf{D}\,\mathsf{Mod}\,A/AeA \longrightarrow \mathsf{D}\,\mathsf{Mod}\,A \longrightarrow \mathsf{D}\,\mathsf{Mod}\,eAe$ setup of [CPS88], set $A = A^{\geq\theta}$ and $e = e^{\theta}$ to get:

$$\mathsf{D}^-\,\mathsf{Mod}\,A^{>\theta} \xrightarrow{\iota_\theta} \mathsf{D}^-\,\mathsf{Mod}\,A^{\geq\theta} \xrightarrow{\jmath^\theta = e^\theta \square} \mathsf{D}^-\,\mathsf{Mod}\,A^\theta$$

$$\iota_\theta^* = A^{>\theta} \overset{\mathsf{L}}{\otimes}_{A^{\geq\theta}} \square \qquad \jmath_!^\theta = A^{\geq\theta} e^\theta \otimes_{A^\theta} \square$$

$$\iota_\theta^! = \bigoplus_i \mathrm{RHom}_{A^{\geq\theta}}(A^{>\theta} 1^i, \square) \qquad \jmath_*^\theta = \bigoplus_i \mathrm{Hom}_{A^\theta}(e^\theta A^{\geq\theta} 1^i, \square)$$

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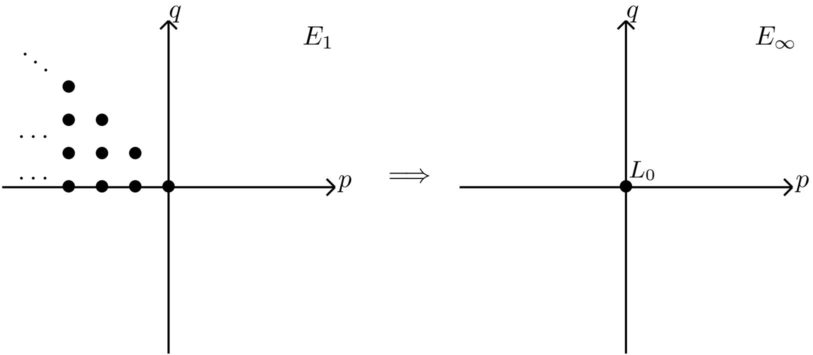

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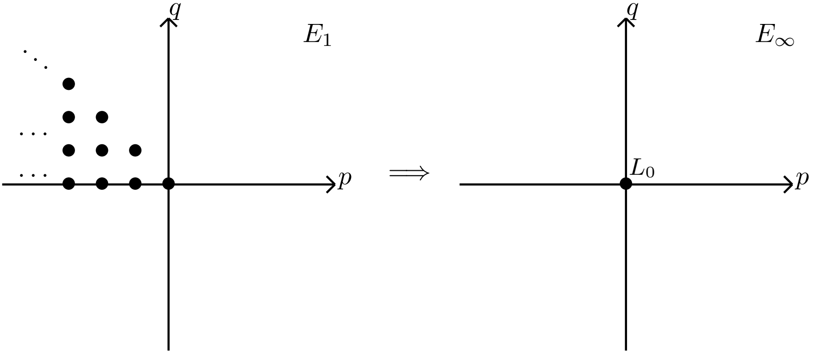

Remark: Cf. Koszul duality.

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Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

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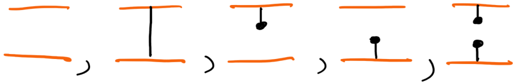
- Then the spectral sequence above exactly recovers the BGG resolution.
- Remark: It can also recover the standard filtration of projectives.

The algebra controlling this

- Recall that the principal block of $\mathcal{O}(\mathfrak{sl}_2)$ is Morita equivalent to the 5-dimensional algebra $A_{\mathfrak{sl}_2}$ spanned by



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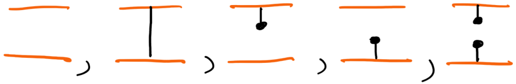
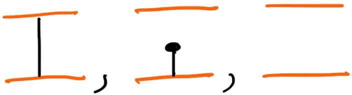


- The subalgebra $A^-_{\mathfrak{sl}_2}$ of this spanned by



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- We wish to mimic the story of Lie algebra cohomology $H^\bullet(\mathfrak{n}^+ : \square)$.

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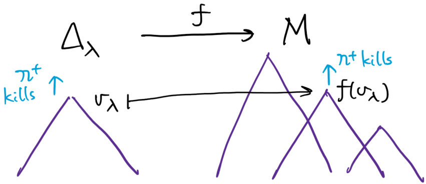

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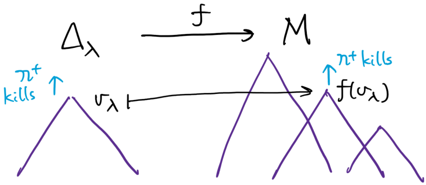

- This can be computed with the Chevalley-Eilenberg complex, which is finite in length.

Nilcohomology

- To this end, try to define

$$A^- := \bigoplus_{\psi \leq \theta} e^\psi \mathbb{C}\mathrm{Y} e^\theta = \mathbb{K} \oplus I,$$

where $\mathbb{K} := \bigoplus_\theta \Bbbk e^\theta$, $I = \bigoplus_{\psi < \theta} e^\psi \mathbb{C}\mathrm{Y}_+ e^\theta$.

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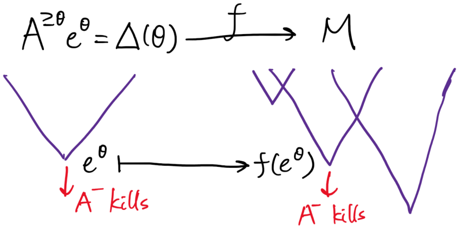

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- Remark: This can be made more uniform by defining

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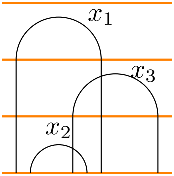

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Theorem

Let $t = 0$ and $\theta = 2n$. Then the dimension of the Ext groups are

$$\dim_q \operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^* = \frac{q^{-n}[2n]!}{(1-q^{-4})\cdots(1-q^{-4n})}$$

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Moreover, the $\mathrm{N}\mathcal{B}^{\theta}$-module $\operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^*$ is isomorphic to a quotient of the polynomial ring,

$$\operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^* \cong \mathbb{C}[X_1, \cdots, X_\theta]\big/\langle p_1, p_3, \cdots, p_{2n-1}\rangle.$$

This proves the theorem:

Theorem (Z. 2024)

At parameter $t = 0$, the 1-dimensional simple L_0 has a BGG resolution

$$\cdots \to C^{-n}_{\mathrm{BGG}}(L_0) \longrightarrow C^{-(n-1)}_{\mathrm{BGG}}(L_0) \longrightarrow \cdots \longrightarrow C^{0}_{\mathrm{BGG}}(L_0) \longrightarrow L_0 \longrightarrow 0$$

where the terms have character

$$\chi(C^{-n}_{\mathrm{BGG}}(L_0)) = \frac{q^{-n}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4n})}\chi(\overline{\Delta}_{2n})$$

and admit filtrations $C^{-n}_{\mathrm{BGG}}(L_0) = \mathrm{F}^0_{\mathrm{BGG}} \supset \mathrm{F}^1_{\mathrm{BGG}} \supset \cdots$ such that

$$\mathrm{gr}^k\, C^{-n}_{\mathrm{BGG}}(L_0) = \overline{\Delta}_{2n} \otimes_{\mathbb{C}} q^{-n}\mathbb{C}[p_2, p_4, \cdots, p_{2n}]_{\deg_{\mathrm{sym}}=k},$$

where $\deg_{\mathrm{sym}} p_i = 1$.

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Question

What do the standard modules correspond to?

Jacobi-Trudi

- The Jacobi-Trudi determinant identity expressed (skew) Schur functions as a determinant of h's:

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Question

Is there a way to witness this as a BGG/highest weight phenomenon?

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- The resulting nilcohomology concentration result will give a BGG resolution categorifying Jacobi-Trudi.

Non-diagrammatic algebras

- Consider (a block of) the degenerate affine Hecke algebra $\widehat{\mathcal{H}}_n$.
- As just a diagrammatic algebra, this has no obvious triangular basis.
- However, we can use Young idempotents to stratify the algebra or module category.
- We can try to define an analog of A^-, which will be Koszul.
- The resulting nilcohomology concentration result will give a BGG resolution categorifying Jacobi-Trudi.
- This approach has the benefit of realizing this phenomenon as a lowest-weight theory; for example, we can realize the permutation module as an actual standardization functor $\jmath_!^\lambda$.

Stratifications within stratifications

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- However, this structure alone does not utilize the obvious ordering on the simples of each Cartan.
- By using the stratification of $\widehat{\mathcal{H}}_n$ above, we should obtain finer stratifications.

Thank you!

Thank you for coming to my talk!
Questions

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Koszul duality

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Here $\operatorname{sh} M = M[n]$ if M is concentrated in Koszul degree n, and $\operatorname{refl}(M)_j = M_{-j}$.

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Then the spectral sequence looks like

$$\Delta \otimes_{A^\circ} \mathcal{K}_{A^+}(\square).$$